EXAM 3 SOLUTION

Problem I (27 Points)

a) $$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(w-k)x}\, dx$$

$$= \frac{1 - e^{i(w-k)2\pi}}{2\pi i\,(k-w)} = \frac{1-e^{i2\pi w}}{2\pi i\,(k-w)}$$

b) c_k has $1/k$ decay rate. It is explained by the jump at 0.

$$\sum_{-\infty}^{+\infty} |c_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} |e^{iwx}|^2 dx = 1$$

c) $$\frac{df}{dx} = \delta(x)\left(1 - e^{i2\pi w}\right) + iw\, e^{iwx}$$

The fourier coefficients of df/dx are given by

$$d_k = ik\, c_k = ik\, \frac{\left(1 - e^{i2\pi w}\right)}{2\pi i(k-w)}$$

$$df/dx = \sum_{-\infty}^{+\infty} ik\, \frac{\left(1 - e^{i2\pi w}\right)}{2\pi i\,(k-w)}\, e^{ikx}$$

Problem 2

a) $w = e^{i2\pi/N}$ with $N = 8$

$w = e^{i\pi/4} = \frac{\sqrt{2}}{2}(1+i)$

We have to check that the inner product:

$\overline{t(\text{col } 1)} \cdot (\text{col } 7) = 0$

$1 + \bar{w}w^7 + \bar{w}^2 w^{14} + \dots + \bar{w}^7 w^{49} = 0$

$\Leftrightarrow 1 + \bar{w}w^7 + (\bar{w}w^7)^2 + \dots + (\bar{w}w^7)^7 = 0$

$\Leftrightarrow 1 + w^6 + w^{12} + \dots + w^{42} = 0$ since $\bar{w}w^7 = w^6$.

$\Leftrightarrow \dfrac{1-(w^6)^8}{1-w^6} = 0$.

This is verified since $w^8 = 1$.

b) $c = F^{-1} f = \frac{1}{N} \bar{F} f$.

the orthogonality shows that row i of $\bar{F}$ times column j of F is $N = 8$ if $i = j$, and 0 if $i \neq j$

so $c = \frac{1}{8}\bar{F}.\text{col } 1 + \frac{1}{8}\bar{F}.\text{col } 7 = \frac{1}{8}\begin{bmatrix}0\\8\\0\\0\\0\\0\\0\\0\end{bmatrix} + \frac{1}{8}\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\8\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\1\end{bmatrix}$

A vector $f = a_0(\text{col } 0) + a_1(\text{col } 1) + \dots + a_7(\text{col } 7)$ will produce a c with a zero component iff one of the a_i is zero.

Problem III

a) $C = \begin{pmatrix} c_0 & c_2 & c_1 \\ c_1 & c_0 & c_2 \\ c_2 & c_1 & c_0 \end{pmatrix}$

b) $c \circledast d = b$

$\Rightarrow F_3(c \circledast d) = F_3(b)$

$\Rightarrow f .* g = h$

$h = F_3 b = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ $\quad f = F_3 c = \begin{bmatrix} c_0 + c_1 + c_2 \\ c_0 + c_1 w + c_2 w^2 \\ c_0 + c_1 w^2 + c_2 w^4 \end{bmatrix}$

$\Rightarrow g = \begin{bmatrix} 3/(c_0 + c_1 + c_2) \\ 0 \\ 0 \end{bmatrix}$

$d = F_3^{-1} g = \frac{1}{3} \overline{F_3} g = \left(\frac{1}{c_0 + c_1 + c_2}, \frac{1}{c_0 + c_1 + c_2}, \frac{1}{c_0 + c_1 + c_2} \right)$

c) $h = f .* g \Rightarrow \begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} h_0 / f_0 \\ h_1 / f_1 \\ h_2 / f_2 \end{bmatrix}$

d) The solution will fail because one of the denominators would be 0.

This correspond to the situation where

$c_0 + c_1 + c_2 = 0 \quad (f_0 = 0)$

or $c_0 + w c_1 + w^2 c_2 = 0 \quad (f_1 = 0)$

or $c_0 + w^2 c_1 + w^4 c_2 = 0 \quad (f_2 = 0)$.

Problem IV

a) $\hat{s}(k) = \int_{-\infty}^{+\infty} s(x) e^{-ikx} dx$

$= \int_{-\infty}^{+\infty} f(x-h) e^{-ikx} dx$. Let $y = x - h$. $dy = dx$

$= \int_{-\infty}^{+\infty} f(y) e^{-ik(y+h)} dy$

$= e^{-ikh} \int_{-\infty}^{+\infty} f(y) e^{-iky} dy = e^{-ikh} \hat{f}(k)$

b) $-\frac{1}{h^2}\left[e^{ikh}\hat{G}(k) - 2\hat{G}(k) + e^{-ikh}\hat{G}(k)\right] + a^2 \hat{G}(k) = 1$

$\hat{G}(k) = \dfrac{h^2}{2 - 2\cos kh + a^2h^2}$

c) As $\int_{-\infty}^{+\infty} \delta(x-1) e^{-ikx} dx = e^{-ik}$,

this would multiply $\hat{G}(k)$ by e^{-ik}.

This corresponds to a shift of the solution (from $G(x)$ to $G(x-1)$).